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Time-dependent Lagrangian systems: a geometric approach using semibasic forms

Manuel F Rañada

Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, 5153475+08Zaragoza, Spain

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Abstract. A new presentation of the geometric formalism in $\mathbb{R} \times TQ$ of the time-dependent Lagrangian systems is given. The formalism is used to prove some properties of the equations determining the Euler-Lagrange vector field X_L, for relating the theory with the time-independent case and for studying the time-dependent Lagrangian inverse problem.

1. Introduction

Nowadays it is known that symplectic geometry is the appropriate geometric setting for the description of autonomous systems in both the Hamiltonian and Lagrangian approaches (see Abraham and Marsden 1978, Marmo *et al* 1985, and references therein). Nevertheless, the Lagrangian formalism seems to be not so straightforward as the Hamiltonian is, because the symplectic structure is not intrinsically defined, but it is L-dependent, i.e. it must be constructed using the Lagrangian function. Time dependence introduces new problems, mainly because one must work in odd-dimensional manifolds. This difficulty is solved by exchanging symplectic forms for contact structures. As a consequence of this, for determining the Euler-Lagrange vector field X_L it will be necessary to supplement the contact equations (which determine not a vector field but a distribution) with some additional condition.

Recently (Cariñena and Rañada 1989) a new presentation of the geometric theory of time-dependent systems has been given for the Hamiltonian formulation. The purpose of this paper is to carry out a similar approach for the time-dependent Lagrangian dynamics. Notice that, although the Hamiltonian formulation can be considered as a previous result, the Lagrangian theory will be directly studied without making use of it, i.e. the theory will be developed in the Lagrangian evolution space, $\mathbb{R} \times TQ$, without making use of the (inverse) Legendre transformation.

2. Evolution space

Let us denote by Q the configuration space of a Lagrangian dynamical system with n degrees of freedom and by $\{q^i;\ i=1,\ldots,n\}$ the local coordinates. Then the evolution space of the system will be $\mathbb{R} \times TQ$ (TQ is the tangent bundle of Q) and the Lagrangian of the system will be a function $L: \mathbb{R} \times TQ \to \mathbb{R}$.

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The natural coordinate on $\mathbb{R}$ will be denoted by t, and $\pi_1: \mathbb{R}\times TQ \to \mathbb{R}$ and $\pi_2: \mathbb{R}\times TQ \to TQ$ are the two projections onto $\mathbb{R}$ and TQ, respectively (for the definition of a time projection in a more general manifold see Lichnerowicz (1976)). The manifold $\mathbb{R}\times TQ$ also possess other vector bundle structures such as $\tau_2: \mathbb{R}\times TQ \to Q$ defined by $\tau_2 = \tau_Q \circ \pi_2$ where $\tau_Q: TQ \to Q$ is the natural bundle projection and $\tau: \mathbb{R}\times TQ \to \mathbb{R}\times Q$ defined by $\tau = \mathrm{Id}_{\mathbb{R}} \times \tau_Q$.

2.1. *Fundamental objects in $\mathbb{R}\times TQ$*

As for the autonomous case, the mathematical formalism for the time-dependent Lagrangian dynamics can be developed mainly using two approaches.

(i) The 'indirect' approach consisting of the study of first the Hamiltonian formalism in $\mathbb{R}\times T^*Q$ and then 'pulling-back' the results to $\mathbb{R}\times TQ$. This approach considers the Hamiltonian formalism as a more fundamental one due to the fact that the cotangent bundle T^*Q of the configuration space Q carries a natural symplectic structure ω_0.

(ii) The tangent bundle geometry has no natural symplectic structure, but instead possesses and induces in $\mathbb{R}\times TQ$ (Crampin *et al* 1984, Cariñena and Martínez 1989) two important geometrical objects: the Liouville vector field Δ and the (1, 1)-tensor field S; so, the 'direct' approach is constructed using the actions of Δ and S.

In terms of local coordinates $\{t, q^i, v^i;\ i = 1, \ldots, n\}$, Δ and S have the form

$$\Delta = v^i\, \partial/\partial v^i \tag{1}$$

$$S = \partial/\partial v^i \otimes (\mathrm{d}q^i - v^i\, \mathrm{d}t). \tag{2}$$

We will use the notation S^* instead of S when it acts on $\Lambda^1(\mathbb{R}\times TQ)$.

2.2. *Product structure of the evolution space*

A vector field defined in $\mathbb{R}\times TQ$ is said to be 'vertical' (with respect to π_2) if its value at any point $(t, m) \in \mathbb{R}\times TQ$ is tangent to the fibre $\pi_2^{-1}(m)$. The product structure of $\mathbb{R}\times TQ$ and the natural chart for $\mathbb{R}$ will permit us to define a vector field $\partial/\partial t$ that gives a basis for the $C^\infty(\mathbb{R}\times TQ)$-module of 'vertical' vector fields. Also, the 1-form $\mathrm{d}t$ defines a $2n$-dimensional distribution that will be called 'horizontal'; thus a 'horizontal' vector field will be a vector field that takes values on that distribution. We will denote by $\mathscr{X}^{\mathrm{H}}(\mathbb{R}\times TQ)$ the set of such vector fields:

$$\mathscr{X}^{\mathrm{H}}(\mathbb{R}\times TQ) = \{X \in \mathscr{X}(\mathbb{R}\times TQ) / \langle \mathrm{d}t, X\rangle = 0\}. \tag{3}$$

In a similar way a 1-form $\alpha \in \Lambda^1(\mathbb{R}\times TQ)$ is called 'semibasic' (with respect to π_2) if the contraction of α with any vertical vector field vanishes, i.e. $\mathrm{i}(\partial/\partial t)\alpha = 0$. We will denote by $\Lambda^1_{\mathrm{sb}}(\mathbb{R}\times TQ)$ the set of such 1-forms:

$$\Lambda^1_{\mathrm{sb}}(\mathbb{R}\times TQ) = \{\alpha \in \Lambda^1(\mathbb{R}\times TQ) / \langle \alpha, \partial/\partial t\rangle = 0\}. \tag{4}$$

Given a 1-form $\alpha \in \Lambda^1(\mathbb{R}\times TQ)$, we will denote by α^{sb} the semibasic part of α, i.e.

$$\alpha^{\mathrm{sb}} = \alpha - [\mathrm{i}(\partial/\partial t)\alpha]\, \mathrm{d}t. \tag{5}$$

In particular, if $F \in C^\infty(\mathbb{R}\times TQ)$, then the semibasic differential of F will be the semibasic 1-form $\mathrm{d}^{\mathrm{sb}}F$ defined by

$$\mathrm{d}^{\mathrm{sb}}F = \mathrm{d}F - [\mathrm{i}(\partial/\partial t)\mathrm{d}F]\, \mathrm{d}t. \tag{6}$$

3. Time-dependent Lagrangian formalism

Suppose that a Lagrangian is given, i.e. a differentiable function L on $\mathbb{R}\times TQ$. Then one can construct a 2-form $\omega_L \in \Lambda^2(\mathbb{R}\times TQ)$ and an energy function $E_L \in C^\infty(\mathbb{R}\times TQ)$ by

$$\omega_L = -d\theta_L \qquad \theta_L = [S^*(dL)]^{sb} \tag{7}$$

$$E_L = \Delta(L) - L. \tag{8}$$

If L is a regular Lagrangian then $(\mathbb{R}\times TQ, \omega_L)$ is a contact manifold; i.e. $\mathbb{R}\times TQ$ is an odd-dimensional manifold and the closed 2-form ω_L is of maximal rank. The proof is as follows. If a vector field Z is

$$Z = a^i(t, q, v)\partial/\partial q^i + b^i(t, q, v)\partial/\partial v^i + c(t, q, v)\partial/\partial t$$

in terms of local coordinates, then $i(Z)\omega_L = 0$ becomes

$$(U_{ki} - U_{ik})a^k - W_{ik}b^k - V_i c = 0 \tag{9a}$$

$$W_{ik}a^k = 0 \tag{9b}$$

$$V_k a^k = 0 \tag{9c}$$

where

$$W_{ik} = \partial^2 L/\partial v^i \partial v^k \qquad U_{ik} = \partial^2 L/\partial v^i \partial q^k \qquad V_i = \partial^2 L/\partial v^i \partial t.$$

So if L is regular, which is to say that the matrix $W = [W_{ik}]$ is non-singular, then the only solution of (9*b*) is the trivial one $a^i = 0$, $i = 1, \ldots, n$. Therefore, in this case, (9*b*) and (9*c*) vanish and (9*a*) determines every one of the n coefficients b^k as linear functions of c. Thus Z must have the form

$$Z = c\tilde{Z} \qquad \tilde{Z} = -W^{ki}V_i\,\partial/\partial v^k + \partial/\partial t$$

with $W^{ji}W_{ik} = \delta^j_k$, and consequently Ker ω_L is one dimensional.

Notice that the concept of contact structure is presented according to Abraham and Marsden (1978); nevertheless, other authors used the terms almost-contact structure and contact structure to denote what are here called contact structure and exact contact structure, respectively; see, e.g., Lichnerowicz (1976) and Albert (1988).

The vector field X_L giving the dynamics is obtained by solving the equation

$$i(X)\Omega_L = 0 \tag{10}$$

where

$$\Omega_L = \omega_L + dE_L \wedge dt. \tag{11}$$

If ω_L is of maximal rank then $(\mathbb{R}\times TQ, \Omega_L)$ is also a contact manifold, i.e. $\dim(\text{Ker}\,\Omega_L) = 1$, and (10) will determine X up to a multiplicative function. The Euler–Lagrange vector field X_L is then determined by adding to (10) the condition $\langle dt, X_L\rangle = 1$, and then the solution will turn out to be a second-order differential (SODE) vector field, i.e.

$$X_L = v^i\partial/\partial q^i + f^i(t, q, v)\partial/\partial v^i + \partial/\partial t. \tag{12}$$

Notice that the notation θ_L has been used (Crampin *et al* 1984, Sarlet 1981, Sarlet and Cantrijn 1981, Cariñena and Martínez 1989) in other papers for the 1-form $S^*(dL)+L\,dt$ that here will be denoted as θ_E and defined by

$$\theta_E=\theta_L-E_L\,dt. \tag{13}$$

This θ_E is such that $\Omega_L=-d\theta_E$.

The above definition for θ_L presents, in particular, two important properties: (i) it has the same coordinate expression as the θ_L used in the time-independent formalism and (ii) the theory will clearly show a great similitude with the time-dependent Hamiltonian formalism, since the θ_L so defined will correspond, when going to $\mathbb{R}\times T^*Q$, to the $\tilde{\theta}_0$ where $\tilde{\theta}_0$ is the pull-back to $\mathbb{R}\times T^*Q$ of the natural 1-form θ_0 defined in T^*Q; i.e. $\theta_L=D_L^*(\tilde{\theta}_0)$ and $\omega_L=D_L^*(\tilde{\omega}_0)$ with $D_L:\mathbb{R}\times TQ\to\mathbb{R}\times T^*Q$ being the Legendre transformation.

Proposition. Let $X\in\mathscr{X}(\mathbb{R}\times TQ)$ satisfy $\langle dt, X\rangle=1$. Then $i(X)\Omega_L=0$ is equivalent to $[i(X)\omega_L]^{sb}=d^{sb}E_L$.

Proof. (i) The property $i(X)\Omega_L=0$ can be decomposed in two equations: one corresponding to the vanishing of the semibasic part and the other to the vanishing of the non-semibasic part:

$$[i(X)\Omega_L]^{sb}=[i(X)\omega_L]^{sb}-d^{sb}E_L=0 \tag{14a}$$

$$i(\partial/\partial t)[i(X)\Omega_L]=i(\partial/\partial t)[i(X)\omega_L]+i(X)\,d^{sb}E_L=0. \tag{14b}$$

Therefore it follows that if $i(X)\Omega_L=0$ then

$$[i(X)\omega_L]^{sb}=d^{sb}E_L.$$

(ii) In order to prove the converse, we will show that (14*b*) is not itself independent, but it is a consequence of (14*a*).

Let us suppose that

$$[i(X)\Omega_L]^{sb}=0.$$

Then

$$i(X)\{i(X)\omega_L-[i(\partial/\partial t)(X)\omega_L]\,dt\}-i(X)\,d^{sb}E_L=0$$

and, using $i(X)i(X)\omega_L=0$ and $i(X)dt=1$, we have

$$i(\partial/\partial t)i(X)\omega_L+i(X)\,d^{sb}E_L=0.$$

Therefore if $[i(X)\omega_L]^{sb}=d^{sb}E_L$ then $i(X)\Omega_L=0$. □

For autonomous systems, the geometric version of the Euler–Lagrange equations can be written in two equivalent ways; i.e. if $L: TQ\to\mathbb{R}$ then a vector field $X\in\mathscr{X}(TQ)$ is a solution of $i(X)\omega_L=dE_L$ if and only if it satisfies $\mathbf{L}_X\theta_L=dL$, where $\mathbf{L}_X$ denotes the Lie derivative with respect to X. The following proposition proves, using the approach of semibasic forms, a similar result for time-dependent systems.

Proposition. Let $X\in\mathscr{X}(\mathbb{R}\times TQ)$ satisfy $\langle dt, X\rangle=1$. Then $i(X)\Omega_L=0$ is equivalent to $(\mathbf{L}_X\theta_L)^{sb}=d^{sb}L$ and $S(X)=0$.

Proof. (i) Let us suppose that $i(X)\Omega_L = 0$, then the 2-form ω_L and the energy function E_L will satisfy

$$i(X)\omega_L \wedge dt = dE_L \wedge dt$$

and taking into account

$$i(X)\omega_L = -i(X)\, d\theta_L \qquad dE_L = d[i(X)\theta_L - L]$$

we get

$$\mathbf{L}_X\theta_L \wedge dt = dL \wedge dt$$

which can be rewritten in a different but equivalent way as

$$(\mathbf{L}_X\theta_L)^{sb} = d^{sb}L.$$

Notice that property $E_L = i(X)\theta_L - L$ is true because X is a SODE vector field.

(ii) Conversely, let us assume $(\mathbf{L}_X\theta_L)^{sb} = d^{sb}L$, then

$$\mathbf{L}_X\theta_L \wedge dt = dL \wedge dt.$$

Now, using $S(X) = 0$ we can write

$$\mathbf{L}_X\theta_L = dE_L + i(X)\, d\theta_L + dL$$

and we arrive at

$$[i(X)\omega_L] \wedge dt = dE_L \wedge dt$$

and the equivalence is proved. □

We finish this section with the following remark: if a 1-form α is such that only involves the dq^i, i.e. $\alpha = \alpha_i(t, q, v)\, dq^i$, then $\mathbf{L}_X\alpha \in \Lambda^1_{sb}$ for any SODE vector field. Consequently the above equation $(\mathbf{L}_X\theta_L)^{sb} = d^{sb}L$ can be written directly as $\mathbf{L}_X\theta_L = d^{sb}L$.

4. The inverse problem

The Helmholtz conditions are a set of conditions (usually presented as four) or equations that permit us to indicate when a given set of second-order ordinary differential equations

$$d^2q^i/dt^2 = f^i(t, q, v) \tag{15}$$

are of Lagrangian class, i.e. that they are the Euler–Lagrange equations for some Lagrangian function $L(t, q, v)$. It can be said that the Helmholtz conditions represent for the Lagrangian evolution a similar result to that represented by the Poisson bracket (PB) theorem (Saletan and Cromer 1971) for the Hamiltonian evolution.

In recent years these conditions have been rewritten in a geometric language. First, in the autonomous case, Crampin (1981, 1983) presented the Helmholtz conditions as the conditions to be satisfied by a symplectic form ω, and later Crampin *et al* (1984) generalised the theory to the time-dependent case as conditions to be satisfied by a contact form Ω (which will correspond to the Ω_L).

The geometrical version of the PB theorem (Cariñena and Rañada 1989) leads to the result that a dynamical vector field Γ defined in $\mathbb{R}\times T^*Q$ represents a time evolution of Hamiltonian class if (i) it preserves the 'horizontal' distribution $\mathscr{X}^H(\mathbb{R}\times T^*Q)$ and (ii) it satisfies $(\mathbf{L}_\Gamma \times \tilde{\omega}_0)^{\text{sb}}=0$ where $\tilde{\omega}_0$ is the pull-back to $\mathbb{R}\times T^*Q$ of the natural symplectic structure ω_0 defined in T^*Q. In the Lagrangian case we are studying now, the first of these two properties becomes trivial since the dynamical vector field X defined in $\mathbb{R}\times TQ$ is supposed to be SODE, and a SODE vector field always preserves $\mathscr{X}^H(\mathbb{R}\times TQ)$. The following theorem studies the inverse problem of the time-dependent Lagrangian mechanics using the language of semibasic forms and, as we will see, it corresponds to the Lagrangian version of the second of the above two properties of the geometrical PB theorem.

Theorem. Let X be a vector field defined in $\mathbb{R}\times TQ$. Then if X satisfies $S(X)=0$ and $\langle \mathrm{d}t, X\rangle=1$, necessary and sufficient conditions for X to be such that it represents an evolution of Lagrangian class are the existence of a 2-form ω on $\mathbb{R}\times TQ$ such that

(c1) ω is closed and such that $\omega^n\wedge \mathrm{d}t$ is a volume element
(c2) $\omega(\partial/\partial v^i, \partial/\partial v^j)=0 \qquad i,j=1,\ldots,n$
(c3) $\omega(\partial/\partial v^i, \partial/\partial t)=0 \qquad i=1,\ldots,n$
(c4) $(\mathbf{L}_X\omega)^{\text{sb}}=0$.

Proof. First notice that (c2) and (c3) can be grouped as the single condition $\omega(V_1, V_2)=0$, where V_1, V_2 denote any coupled τ-vertical vector fields.

Condition (c1) implies the local existence of a 1-form ψ such that $\omega=\mathrm{d}\psi$. Clearly this 1-form will not be unique, so we will denote by Λ^1_ω the set of such 1-forms:

$$\Lambda^1_\omega=\{\psi\in\Lambda^1(\mathbb{R}\times TQ)/\mathrm{d}\psi=\omega\}.$$

For the proof of the theorem we need the following lemma.

Lemma. Let ω be a 2-form that satisfies the conditions (c1), (c2) and (c3). Then in the associated set Λ^1_ω there is at least a 1-form, which will be denoted by θ, such that

(i) $\mathrm{i}(\partial/\partial v^i)\theta=0 \qquad i=1,\ldots,n$
(ii) $\mathrm{i}(\partial/\partial t)\theta=0$.

Proof. Let ψ be any arbitrary 1-form in Λ^1_ω with a local expression

$$\psi=\alpha_i(t,q,v)\,\mathrm{d}q^i+\beta_i(t,q,v)\,\mathrm{d}v^i+\gamma(t,q,v)\,\mathrm{d}t.$$

Then conditions (c2) and (c3) mean that the coefficients β_i and γ must satisfy the following relations:

$$\partial\beta_i/\partial v^j=\partial\beta_j/\partial v^i \qquad \partial\beta_i/\beta t=\partial\gamma/\partial v^i$$

and consequently they must take the values

$$\beta_i=\partial f^1/\partial v^i+\tilde{\beta}_i(q) \qquad \gamma=\partial f^1/\partial t+\tilde{\gamma}(t,q)$$

where $f^1=f^1(t,q,v)$ is a differentiable function. Therefore the coordinate expression for any ψ in Λ^1_ω will have the form

$$\psi=\alpha_i\,\mathrm{d}q^i+(\partial f^1/\partial v^i+\tilde{\beta}_i)\,\mathrm{d}v^i+(\partial f^1/\partial t+\tilde{\gamma})\,\mathrm{d}t.$$

Now let us denote by θ the 1-form defined by

$$\theta = \psi - \mathrm{d}f$$

where the function f is given by $f = f^1 + f^2 + f^3$ with

$$f^2 = \int \tilde{\gamma}(t, q)\,\mathrm{d}t \qquad f^3 = \tilde{\beta}_i(q)v^i.$$

This new 1-form is such that $\mathrm{d}\theta = \mathrm{d}\psi = \omega$, and besides it satisfies $\mathrm{i}(\partial/\partial t)\theta = 0$ and $\mathrm{i}(\partial/\partial v^i)\theta = 0$, $i = 1, \ldots, n$. □

The proof of the lemma is now done, so we can proceed to the theorem.

First notice that if $\omega^n \wedge \mathrm{d}t \neq 0$ then a vector field X such that $S(X) = 0$ and $\langle \mathrm{d}t, X\rangle = 1$ cannot be a characteristic vector field of ω, i.e. $i(X)\omega \neq 0$.

The condition (c4) can be rewritten as

$$[\mathrm{d}(\mathsf{L}_X\theta)]^{\mathrm{sb}} = 0$$

or, in an equivalent way,

$$\mathrm{d}(\mathsf{L}_X\theta) = \delta \wedge \mathrm{d}t$$

where δ must be such that $\delta \wedge \mathrm{d}t$ is closed.

Let α be a 1-form such that $\alpha \wedge \mathrm{d}t$ is closed, then it follows that $\alpha = \alpha_1 + \alpha_2$ with α_1 and α_2 such that $\mathrm{d}\alpha_1 = 0$ and $\mathrm{d}\alpha_2 = \beta \wedge \mathrm{d}t$. In the case of δ this implies the existence of functions L and z such that

$$\mathsf{L}_X\theta = \mathrm{d}L + z\,\mathrm{d}t.$$ □

Remarks. (i) Actually $\alpha_1 = \mathrm{d}F$ and $\alpha_2 = \beta(t, q, v)\,\mathrm{d}t + \gamma_i(t)\,\mathrm{d}q^i + \zeta_i(t)\,\mathrm{d}v^i$ in such a way that $L = F + \gamma_i q^i + \zeta_i v^i$ and $z = \beta - \gamma_i' q^i + \zeta_i' v^i$.

(ii) Recall that if a 1-form α is such that it only involves the $\mathrm{d}q^i$, i.e. $\alpha = \alpha_i\,\mathrm{d}q^i$, then $\mathsf{L}_X\alpha \in \Lambda^1_{\mathrm{sb}}$. Thus the two functions z and L must be related by $z = -\partial L/\partial t$.

We have arrived at the existence of a function L such that $(\mathsf{L}_X\theta)^{\mathrm{sb}} = \mathrm{d}^{\mathrm{sb}}L$. The function L turns out to be a (local) Lagrangian for the dynamical vector field X. Indeed we will show that the 1-form θ, whose existence is guaranteed by the previous lemma, is really the 1-form θ_{L} associated with L.

A general property of the 1-forms involving only the $\mathrm{d}q^i$, i.e. $\alpha = \alpha_i\,\mathrm{d}q^i$, is that they satisfy

$$\alpha = [S^*(\mathsf{L}_Y\alpha)]^{\mathrm{sb}}$$

for any vector field Y such that $\mathrm{i}(\partial/\partial q^i)Y = v^i$, $i = 1, \ldots, n$; so using this for the θ and the X we have

$$\begin{aligned}\theta &= [S^*(\mathsf{L}_X\theta)]^{\mathrm{sb}} \\ &= [S^*(\mathrm{d}L)]^{\mathrm{sb}}\end{aligned}$$

which completes the proof.

As a final comment let us notice that, once we know the ω, then the associated Ω such that $\mathrm{i}(X)\Omega = 0$ is given by $\Omega = \omega + \mathrm{i}(X)\omega \wedge \mathrm{d}t$.

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